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Noncommutative Yang-Mills from equivalence of star products

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Abstract. It is shown that the transformation between ordinary and noncommutative Yang-Mills theory as formulated by Seiberg and Witten is due to the equivalence of certain star products on the D-brane world-volume.

1 Introduction

The noncommutativity of coordinates in D-brane physics has lately received considerable attention. See [1] and references therein, in particular [2–6]. It was examined thoroughly from different points of view. On one side the transverse coordinates of N coinciding D-branes are described by $N \times N$ matrices on the other side the end points of an open string become noncommutative in the presence of a constant B-field. We shall not go into details here and just mention a fact that is most relevant to the present letter: In both situations D-branes in the presence of a large background gauge field can be equivalently described by either commutative or noncommutative gauge fields.

In this letter we will consider the problem from the D-brane world-volume perspective. The idea is the following: We formulate the problem within the framework of symplectic geometry and Kontsevich's deformation quantization to obtain abstract but general results independent of particularities of specific (path integral) quantizations [7–9]. An equivalence of certain star products will lead us to a transformation between two quantities, which physically can be interpreted as ordinary and non-commutative Yang-Mills fields. Within this approach an existence of such a relation is a priori guaranteed. We then show that such a transformation is necessarily identical to the transformation proposed by Seiberg and Witten [1]. All this can be done rigorously. In the last part of the letter we will discuss how all this is related to the formulation that uses a path integral representation of boundary states [7,8].

2 Classical description

For the classical description of the problem the following lemma of Moser [10] is crutial. Let M be a symplectic manifold and $\omega = \omega_{ij}(x)\mathrm{d}x^i \wedge \mathrm{d}x^j$ the symplectic form on M. The symplectic form is closed $\mathrm{d}\omega = 0$ and its coefficient matrix nondegenerate $\det \omega_{ij}(x) \neq 0$ for all $x \in M$. If ω' is another symplectic form on M such that it belongs to the same cohomology class as ω and if the t-dependent form ($t \in [0,1]$)

$$\Omega = \omega + t(\omega' - \omega) \tag{1}$$

is nondegenerate, then

$$\omega' - \omega = \mathrm{d}a \tag{2}$$

for some 1-form a, the t-dependent vector field X, implicitly given by

$$i_X \Omega + a = 0 \tag{3}$$

is well defined and

$$\mathcal{L}_X \Omega \equiv \mathrm{d}(i_X\Omega) + i_X \mathrm{d}\Omega + \partial_t \Omega = -\mathrm{d}a + (\omega' - \omega) = 0. \tag{4}$$

This implies that all $\Omega(t)$ are related by coordinate transformations generated by the flow of X: $\rho^*_{tt'}\Omega(t') = \Omega(t)$, where $\rho^*_{tt'}$ is the flow of X. Setting $\rho^* = \rho^*_{01}$ we have in particular

$$\rho^* \omega' = \omega. \tag{5}$$

Explicitely

$$\rho^* = e^{\partial_t + X} e^{-\partial_t}\big|_{t=0} = e^{\theta^{ij} a_j \partial_i - \frac{1}{2}\theta^{ik} f_{kl} \theta^{lj} a_j \partial_i + o(\theta^3)}, \tag{6}$$

where $\theta^{ij}\omega_{jk} = \delta^i_k$ and $f_{kl} = \partial_k a_l - \partial_l a_k$.

The only complication is that X may not be complete, which is no problem for M compact. For noncompact M (in our case an open domain in $\mathbb{R}^{2n}$) we have to treat t as a formal parameter and work with formal diffeomorphisms given by formal power series in t. Specifying $t = 1$ amounts to considering formal power series in the matrix elements of $(\omega' - \omega)$. This is the same as assuming that $\mathrm{d}a$ is small or ω large. Alternatively we could work with formal power series in $\theta^{ij} = \omega^{-1}_{ij}$. In either case Ω is nondegenerate. In this sense we always have a coordinate change on M

which relates the two symplectic forms ω and ω'. In the cases $t = 0$ and $t = 1$ we denote the Poisson brackets by $\{,\}$ and $\{,\}'$ respectively.

Consider now a gauge transformation $a \mapsto a + \mathrm{d}\lambda$. The effect upon X will be

$$X \mapsto X + X_\lambda, \tag{7}$$

where X_λ is the Hamiltonian vector field

$$i_{X_\lambda}\Omega + \mathrm{d}\lambda = 0 \tag{8}$$

and $\mathcal{L}_{X_\lambda}\Omega = 0$. The whole transformation induced by λ, including the coordinate transformation ρ^* corresponding to a is

$$f \overset{(\lambda)}{\mapsto} f + \{\tilde\lambda, f\}' \overset{(a)}{\mapsto} \rho^* f + \{\rho^*\tilde\lambda, \rho^* f\}, \tag{9}$$

where we have used $\rho^*\{\tilde\lambda, f\}' = \{\rho^*\tilde\lambda, \rho^* f\}$. We shall give an expression of $\tilde\lambda$ in the case of constant θ in Sect. 4.

Physically we can view the above coordinate transformations either as active or passive, i.e. we either have two different symplectic structures ω, ω' on the same manifold related by an active transformation or we have just one symplectic structure expressed in different coordinates. The additional infinitesimal canonical transformation does not change the symplectic structure. Let us mention that the paper [11] is in fact an explicit realization of the Moser lemma in the situation describing a D-brane in the background gauge field.

3 Deformation quantization

We would now like to consider the deformation quantization [12] of the two symplectic structures ω and ω' a la Kontsevich. We follow the definitions and conventions of [13].

The set of equivalence classes of Poisson structures on a smooth manifold M depending formally on $\hbar$,

$$\alpha(\hbar) = \alpha_1\hbar + \alpha_2\hbar^2 + \ldots, \qquad [\alpha, \alpha] = 0, \tag{10}$$

where $[,]$ is the Schouten-Nijenhuis bracket of polyvector vector fields, is defined modulo the action of the group of formal paths in the diffeomorphism group of M, starting at the identity diffeomorphism. Within the framework of Konstevich's deformation quantization the equivalence classes of Poisson manifolds can be naturally identified with the sets of gauge equivalence classes of star products on the smooth manifold M. The Poisson structures α, α' can be identified with the series $\alpha(\hbar) = \hbar\alpha$ and $\alpha'(\hbar) = \hbar\alpha'$, and via Kontsevich's construction with canonical gauge equivalence classes of star products. In view of Moser's Lemma the resulting star products will also be equivalent in the sense of deformation theory.

Since the two star products $*$ and $*'$ on M, corresponding to $\alpha(\hbar)$ and $\alpha'(\hbar)$, are equivalent, there exists an automorphism $D(\hbar)$ of $A[[\hbar]]$, which is a formal power series in $\hbar$, starting with the identity, with coefficients that are differential operators on $A \equiv C^\infty(M)$, such that for any two smooth functions f and g on M

$$f(\hbar) *' g(\hbar) = D(\hbar)^{-1}(D(\hbar)f(\hbar) * D(\hbar)g(\hbar)). \tag{11}$$

Note, that we first have to take care of the classical part of the transformation via pullback by ρ^*, so that the remaining automorphism $D(\hbar)$ is indeed the identity to zeroth order in $\hbar$. The complete map, including the coordinate transformation is $\mathcal{D} = D(\hbar) \circ \rho^*$.

The inner automorphisms of $A[[\hbar]]$, given by similarity transformation

$$f(\hbar) \mapsto \Lambda(\hbar) * f(\hbar) * (\Lambda(\hbar))^{-1}, \tag{12}$$

with invertible $\Lambda(\hbar) \in A[[\hbar]]$, do not change the star product. Infinitesimal transformations that leave the star product invariant are necessarily derivations of the star-product. The additional gauge transformation freedom $A \to A + d\lambda$ in Moser's lemma induces an infinitesimal canonical transformation and, after quantization, an inner derivation of the star product $*'$. We will use the fact that this transformation (including classical and quantum part) can be chosen as

$$f \mapsto f + i\tilde\lambda *' f - if *' \tilde\lambda. \tag{13}$$

This, as we shall see, directly lead to the celebrated relation of a noncommutative gauge transformation.

We shall not try to review deformation quantization and Kontsevich's formula for the star product in its full generality here. A detailed description is given in the original paper [13], the path integral representation using a topological sigma-model on the disc was developed in [14] and an excellent historical overview of deformation quantization and many references can be found in [15]. We only note that in the case of constant Poisson tensor α one obtains the well-known Moyal bracket. We are, however, interested in the existence of a natural star product for any Poisson manifold, which is guaranteed according to Kontsevitch. In this letter we technically only use the corresponding result for symplectic manifolds [16–18].

In the following we will absorb $\hbar$ in θ, θ', etc.

4 Seiberg-Witten transformation

To make contact with the discussion of Seiberg and Witten we take ω to be the symplectic form on $\mathbb{R}^{2n}$, the D-brane world-volume, induced by a constant B-field:

$$\omega = \theta^{-1}_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j \tag{14}$$

with

$$\theta^{ij} = \left(\frac{1}{g+B}\right)^{ij}_{A}; \tag{15}$$

g is the constant closed string metric and the subscript A refers to the antisymmetric part of a matrix (we have set $2\pi\alpha' = 1$). In the zero slope limit

$$\omega = B. \tag{16}$$

For ω' we take

$$\omega' \equiv (\theta')^{-1}_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j = \omega + F, \qquad (17)$$

where $F = F_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j$ is the field strength of the rank one gauge field A. (The extension of the following to higher rank is straightforward.) We are in the situation of Sect. 2, with $a = A$ being the gauge field. The star products induced by Poisson structures θ and θ' are equivalent and the equivalence transformation (including the classical part) is given by the map $\mathcal{D} = D \circ \rho^*$, where $\rho^* = \mathrm{id} + \theta^{ij}A_j\partial_i + \frac{1}{2}\theta^{kl}A_l\partial_k\theta^{ij}A_j\partial_i + \frac{1}{2}\theta^{kl}\theta^{ij}A_lF_{kj}\partial_i + o(\theta^3)$, see also [11]; D acts trivially on x^i to this order, but can of course in principle be computed order by order to any order in θ. It is convenient to write the result of $\mathcal{D}$ acting on the coordinate functions x^i in the form [11,7,8]

$$\begin{aligned}\mathcal{D}x^i &= x^i + \theta^{ij}\hat{A}_j \\ &= x^i + \theta^{ij}A_j + \frac{1}{2}\theta^{kl}\theta^{ij}A_l(\partial_k A_j) \\ &\quad + \frac{1}{2}\theta^{kl}\theta^{ij}A_lF_{kj} + o(\theta^3) \qquad (18)\end{aligned}$$

with $\hat{A}$ a function of x depending on θ, A and derivatives of A, as shown. It is obvious that $\hat{A}$ has the form $\hat{A} = A + o(\theta) + \ldots$, since to lowest order in θ it has to reproduce the coordinate transformation ρ^* relating the two symplectic forms ω and ω'.

Let us now discuss what effect a gauge transformation $A \mapsto A + d\lambda$ has in this picture: It represents the freedom in the choice of symplectic potential $A' = \frac{1}{2}\omega_{ji}x^j\mathrm{d}x^i + A$ for ω'. In Sect. 2 we found that classically the gauge transformation amounts to an infinitesimal canonical transformation, and, after deformation quantization, it has the form (13). The whole map is

$$f \overset{(\lambda)}{\mapsto} f + i\tilde{\lambda} *' f - if *' \tilde{\lambda} \overset{(A)}{\mapsto} \mathcal{D}f + i\mathcal{D}\tilde{\lambda} * \mathcal{D}f - i\mathcal{D}f * \mathcal{D}\tilde{\lambda}. \qquad (19)$$

Let us introduce $\hat{\lambda}$ as a shorthand for $\mathcal{D}\tilde{\lambda}$. $\hat{\lambda}$ obviously depends on θ, A, derivatives of A and the classical gauge transformation λ. Explicitely:

$$\begin{aligned}\tilde{\lambda} &= \lambda - \frac{1}{2}\theta^{ij}A_j(\partial_i\lambda) + o(\theta^2), \\ \hat{\lambda} &= \lambda + \frac{1}{2}\theta^{ij}A_j(\partial_i\lambda) + o(\theta^2). \qquad (20)\end{aligned}$$

We would like to express the result of the map (19) acting on the coordinates x^i again in the form (18), but with $\hat{A}_j$ replaced with $\hat{A}_j + \delta\hat{A}_j$. Using (18) and $x^i * x^j - x^j * x^i = i\theta^{ij}$ to compute the $*$-commutator $[\hat{\lambda} \overset{*}{,} x^i]$, we find

$$\delta\hat{A}_i = \partial_i\hat{\lambda} + i\hat{\lambda} * \hat{A}_i - i\hat{A}_i * \hat{\lambda}. \qquad (21)$$

We see, as expected, that the relation between $\hat{A}$ and A implied by the coordinate transformation (18) is precisely the same as the one proposed by Seiberg and Witten based on the expectation that an ordinary gauge transformation on A should induce a noncommutative gauge transformation (21) on $\hat{A}$. We furthermore see that within the framework of deformation quantization a la Kontsevich the existence of such a transformation between the commutative and noncommutative descriptions is guaranteed. It is not hard to compute the terms of higher order in θ directly in our approach.

In essence the Seiberg-Witten transformation between the commutative and noncommuative description of D-branes is possible due to equivalence of two star products, namely the one defined by the Poisson tensor θ (15) and the another one defined by the Poisson θ' (17).

Let us remark that we make contact here with another approach to noncommutative gauge theory [19], whose relations and manipulations resemble the ones of this section, but with a different philosophy. Equation (18) defines a covariant coordinate in that theory.

5 Relation to boundary states formalism

The string boundary state coupled to the $U(1)$ gauge field admits a path integral representation. Let $|D\rangle$ be a Dirichlet boundary state, $X^i(\sigma)|D\rangle = 0$ at some fixed instant $\tau = 0$. X^i are the string coordinates and P_i the conjugate momenta. The boundary state $|B\rangle$ coupled to a U(1) gauge field A' is then given as

$$|B\rangle = \int \mathcal{D}x \exp(i\int \mathrm{d}\sigma A'_i(x)\partial_\sigma x^i - P_i x^i)|D\rangle. \qquad (22)$$

The path integral itself can be interpreted within the framework of Kontsevich deformation quantization [14]: If we denote $*'$ the star product obtained a la Kontsevich from $\omega' = \mathrm{d}A'$ then the path integral is the trace of the path-ordered exponential $\mathrm{P}[\exp(-i\int \mathrm{d}\sigma P_i x^i)]_{*'}$, where we assume implicitly the $*'$-product within the exponential. In the notation of the previous section $A' = \frac{1}{2}B_{ij}x^i\partial_\sigma x^j + A$. Let us translate the gauge equivalence of the star products $*'$ and $*$ to the language of boundary states. We get the condition

$$\begin{aligned}&\int \mathcal{D}x\ \exp\left(i\int \mathrm{d}\sigma\frac{1}{2}B_{ij}x^i\partial_\sigma x^j + A - P_i x^i\right)|D\rangle \qquad (23)\\ &= \int \mathcal{D}x\ \exp\left(i\int \mathrm{d}\sigma\frac{1}{2}B_{ij}x^i\partial_\sigma x^j - P_i(x^i + \theta^{ij}\hat{A}_j)\right)|D\rangle.\end{aligned}$$

This is exactly the condition of [8]. The above equality is evidently true even without path integrals on its both sides acting on the Dirichlet boundary state.

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